Bi-Hamiltonian formulation of the O(3) chiral fields equations hierarchy via a polynomial bundle

1998 J. Phys. A: Math. Gen. 31 8709

(http://iopscience.iop.org/0305-4470/31/43/013)

J. Phys. A: Math. Gen. **31** (1998) 8709–8726. Printed in the UK PII: S0305-4470(98)89283-4

Bi-Hamiltonian formulation of the O(3) chiral fields equations hierarchy via a polynomial bundle

A B Yanovski

Faculty of Mathematics and Informatics, St. Kliment Ohridski University, James Bouchier Boul. 5, 1164—Sofia, Bulgaria

Received 17 November 1997, in final form 6 May 1998

Abstract. We investigate the Hamiltonian structures of some hierarchy of evolution equations related to a polynomial bundle over the algebra $sl(4)$. The bundle contains the polynomial Lax pair for the O(3) chiral fields system and for that reason the corresponding hierarchy of nonlinear evolution equations is called the CF hierarchy. It is known that the Hamiltonian properties of the CF hierarchy may be explained as a consequence of the existence of compatible Poisson structures arising from a different Lie algebra structure defined over $so(4)$. We show that the generating operator for the CF hierarchy can be regarded as a Nijenhuis tensor on the manifold of potentials and then naturally this manifold is equipped with the Poisson–Nijenhuis structure.

1. Introduction

It is known that the discovery of the inverse scattering method, (see for example the monograph book [6]), permitted us to apply a number of classical results from the spectral theory of the operators to the problem of obtaining exact solutions for some special class of evolution equations depending on one spatial variable x, now called soliton equations. In the past decades the evolution of the original ideas gave rise to a number of various approaches to the investigation of the soliton equations and the remarkable properties of these equations attracted general interest. Nowadays we have a number of approaches treating not only the questions of how to find exact solutions of the soliton equations but also the algebraical and geometrical theory for the Hamiltonian structures of the soliton equations, the existence of infinite series of conservation laws and other related topics which are difficult even to list. However, there is one property that is characteristic of the soliton equations and although it is interpreted in different ways in different theories it plays a crucial role in all of the approaches—the soliton equations can be written as the compatibility condition between two linear operators L and M:

$$[L, M] = 0. \qquad (1)$$

(This representation is called Lax representation and the couple L, M-Lax pair.) Usually the operators L and M have the form

$$L \equiv \frac{\partial}{\partial x} - U(x,t,\lambda) \qquad M \equiv \frac{\partial}{\partial t} - V(x,t,\lambda) \qquad (2)$$

where U, V are some matrix-valued functions, depending on the time t and the spatial variable x through a set of 'potential' or 'field' functions

$$(f_1(x,t), f_2(x,t), \ldots, f_n(x,t)) \equiv f$$

and on the spectral parameter λ. The soliton equation (or system of equations) corresponding to the pair L, M is written in terms of the potential functions and has the form

$$(f_i)_t = F_i(f, f_x, \ldots) \qquad i = 1, 2, \ldots, n.$$

The question about the criteria for the existence of a Lax pair for a given evolution equation (or a system of equations) and the construction of such pairs is very interesting and important. There are some results in this direction, but the problem is far from its final solution. Meanwhile, examples of quite different Lax pairs (not differing only on the representation of the finite-dimensional algebras to which the coefficients $U(x, t, \lambda)$, $V(x, t, \lambda)$ belong) for the same equation are found. For example, in a recent work [1] we introduced new Lax pairs, polynomial in the spectral parameter, for two important physical systems—the system of O(3) chiral fields (CF) equations and the famous Landau–Lifshitz (LL) equation [18]. The pairs that were known before depend on the spectral parameter λ through elliptic functions [3, 4, 33]. Thus, there arises a number of interesting questions about the equivalence of the Lax pairs or in other words whether using different Lax pairs one can obtain the same results. It seems that the problem can be divided into two other problems.

- The equivalence of the hierarchies of evolution equations, the hierarchies of their conservation laws, and the geometrical properties of the corresponding hierarchies.
- The equivalence of the methods for constructing the exact solutions.

In the most trivial case, when the different Lax pairs correspond to different faithful representations of the same algebra, the first of these problems is trivially answered. However, even in this case the second problem is not so simple, as it can be shown that the spectral properties of the corresponding operators actually *depend* upon the representation one works in, see for example [15]. However, in our opinion far more interesting is the case where the Lax pairs differ not only by the choice of the corresponding representations but when the dependence on the spectral parameter in the Lax pairs is different. Of course, in order to answer the questions about the equivalence one must investigate thoroughly the results obtained via different Lax pairs. The elliptic pairs for the LL equation and the O(3) CF equation have attracted general attention and the literature treating the corresponding equations is large enough. Using the elliptic pair it was shown that the LL equation and the system of O(3) CF equations are completely integrable Hamiltonian systems, see [33, 27], and the hierarchies of equations related to the LL and CF equations as well as their Hamiltonian structures are investigated in [7, 8, 10, 32].

There is one additional trend which was initially one of our principal motivations for the search of polynomial Lax pairs for the LL and CF equations. It is well known that when some parameters tend to zero the LL equation transforms into the famous Heisenberg ferromagnet (HF) equation. The auxiliary linear problem for the HF equation is polynomial in λ and one of the well known facts from the theory of the soliton equations is the gauge equivalence between this problem and the Zakharov–Shabat linear problem. From here follows the equivalence between HF and the nonlinear Schrödinger equation [37]. This fact gave rise to the gauge-covariant theory of the generating (recursion) operators [13–15] and it is possible that the new pairs will help us to perform a similar program in the case of LL and CF equations.

As already mentioned, the polynomial pairs for the LL and CF equations have recently been introduced and the theory for them is not so well developed as for the elliptic pairs. In [2] we began the investigation of the LL and CF hierarchies of soliton equations corresponding to the polynomial bundle and have obtained an algorithmic procedure through which one can calculate these hierarchies. In this work we shall investigate more closely the Hamiltonian properties of the CF hierarchy and the corresponding recursion operators which in the geometric approaches are called Nijenhuis tensors.

2. The CF hierarchy of integrable equations

We must recall some facts about the hierarchies of soliton equations, which can be obtained via the $sl(4)$ polynomial bundle, see [2]. In order to do so let us define the following maps from $\mathbb{C}^3$ into the algebra $so(4)$ (the algebra of skew-symmetric 4×4 matrices):

$$\boldsymbol{u} \to \{\boldsymbol{u}\}_I = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tag{3}$$

$$\boldsymbol{v} \to \{\boldsymbol{v}\}_{II} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix}. \tag{4}$$

It can be seen that every element $A \in so(4)$ can be written in the form

$$A = \{\boldsymbol{u}\}_I + \{\boldsymbol{v}\}_{II} \tag{5}$$

and this representation corresponds to the well known splitting of $so(4)$ into direct sum of two $so(3)$ algebras.

The $sl(4)$ bundle we are speaking about consists of the following hierarchy of Lax pairs:

$$L \equiv \frac{\partial}{\partial x} - U \qquad M_N \equiv \frac{\partial}{\partial t} - V_N \qquad U(\lambda) = \tfrac{1}{2}A(\lambda + J) \tag{6}$$

$$V_N(\lambda) = \tfrac{1}{2}(\lambda^N B_0 + \lambda^{N-1}B_1 + \cdots + B_N)(\lambda + J) \qquad N = 0, 1, 2, \ldots \tag{7}$$

where

$$\begin{aligned} A &= \{\boldsymbol{u}\}_I + \{\boldsymbol{v}\}_{II} \\ B_n &= \{\boldsymbol{b}_n\}_I + \{\boldsymbol{c}_n\}_{II} \end{aligned} \tag{8}$$

J is the diagonal matrix

$$J = \begin{pmatrix} -j_1 - j_2 + j_3 & 0 & 0 & 0 \\ 0 & -j_1 + j_2 - j_3 & 0 & 0 \\ 0 & 0 & j_1 - j_2 - j_3 & 0 \\ 0 & 0 & 0 & j_1 + j_2 + j_3 \end{pmatrix} \tag{9}$$

and $\boldsymbol{u}(x,t), \boldsymbol{v}(x,t) \in \mathbb{R}^3$ are smooth vector fields taking values on the unit sphere:

$$(\boldsymbol{u})^2 = 1 \qquad (\boldsymbol{v})^2 = 1. \tag{10}$$

The vector fields $\boldsymbol{u}(x,t), \boldsymbol{v}(x,t)$ obey the following boundary conditions

$$\begin{aligned} &\lim_{x\to\pm\infty} \boldsymbol{u} = \boldsymbol{u}_0 = \text{constant} \\ &\lim_{x\to\pm\infty} \boldsymbol{v} = \boldsymbol{v}_0 = \text{constant} \\ &\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \boldsymbol{u} = 0 \\ &\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \boldsymbol{v} = 0 \\ &n = 1, 2, \ldots \end{aligned} \tag{11}$$

which mean that they converge fast enough to some limit values for $|x| \to \infty$. Let us denote the set of the matrices of the type (8) with $\boldsymbol{u}(x), \boldsymbol{v}(x)$ obeying (10), (11) by $\mathcal{M}$. In other words $\mathcal{M}$ is the set of potentials.

In [2] we proved that the nonlinear evolution equations, corresponding to the hierarchy of Lax pairs introduced above have the following matrix form:

$$A_t = (B_N)_x - \tfrac{1}{2}(AJB_N - B_NJA) \tag{12}$$

and can be written in an equivalent 'vector' form

$$\begin{aligned} &\boldsymbol{u}_t = -\boldsymbol{u} \times \boldsymbol{b}_{N+1} \\ &\boldsymbol{v}_t = -\boldsymbol{v} \times \boldsymbol{c}_{N+1} \\ &N = 0, 1, 2, \ldots \end{aligned} \tag{13}$$

where the functions $\boldsymbol{b}_n, \boldsymbol{c}_n, n = 0, 1, \ldots$ are the solutions of the infinite system:

$$\begin{aligned} &\boldsymbol{u} \times \boldsymbol{b}_0 = 0 \qquad \boldsymbol{v} \times \boldsymbol{c}_0 = 0 \\ &\boldsymbol{u} \times \boldsymbol{b}_{n+1} = -(\boldsymbol{b}_n)_x - K(\boldsymbol{v} \times \boldsymbol{c}_n) + \boldsymbol{u} \times K(\boldsymbol{c}_n) - \boldsymbol{b}_n \times K(\boldsymbol{v}) \\ &\boldsymbol{v} \times \boldsymbol{c}_{n+1} = -(\boldsymbol{c}_n)_x - K(\boldsymbol{u} \times \boldsymbol{b}_n) + K(\boldsymbol{u}) \times \boldsymbol{c}_n - K(\boldsymbol{b}_n) \times \boldsymbol{v} \\ &n = 0, 1, \ldots, N-1. \end{aligned} \tag{14}$$

We call this system the CF chain system. In the above expression K is the diagonal matrix

$$K = \mathrm{diag}(j_1, j_2, j_3)$$

and $(K(\boldsymbol{a}))_i \equiv j_i a_i$. The next proposition gives an algorithm for obtaining successively the functions $\boldsymbol{b}_n, \boldsymbol{c}_n, n = 0, 1, \ldots$.

Proposition 2.1. The CF chain system has the following solution:

$$\begin{aligned} &\boldsymbol{b}_0 = \epsilon \boldsymbol{u} \qquad \boldsymbol{c}_0 = \mu \boldsymbol{v} \\ &\boldsymbol{b}^u_{n+1} = \boldsymbol{u} \times (\boldsymbol{b}^u_n)_x + \langle \boldsymbol{u}, \boldsymbol{b}_n \rangle \boldsymbol{u} \times \boldsymbol{u}_x + [K(\boldsymbol{c}_n^{\,v})]^u - (\langle \boldsymbol{u}, \boldsymbol{b}_n \rangle - \langle \boldsymbol{v}, \boldsymbol{c}_n \rangle)[K(v)]^u \\ &\qquad\qquad + \boldsymbol{u} \times K(\boldsymbol{v} \times \boldsymbol{c}_n^{\,v}) + \langle \boldsymbol{u}, K(\boldsymbol{v}) \rangle \boldsymbol{b}^u_n \\ &\boldsymbol{c}^{\,v}_{n+1} = \boldsymbol{v} \times (\boldsymbol{c}_n^{\,v})_x + \langle \boldsymbol{v}, \boldsymbol{c}_n \rangle \boldsymbol{v} \times \boldsymbol{v}_x + [K(\boldsymbol{b}^u_n)]^v + (\langle \boldsymbol{u}, \boldsymbol{b}_n \rangle - \langle \boldsymbol{v}, \boldsymbol{c}_n \rangle)[K(u)]^v \\ &\qquad\qquad + \boldsymbol{v} \times K(\boldsymbol{u} \times \boldsymbol{b}^u_n) + \langle \boldsymbol{u} K(\boldsymbol{v}) \rangle \boldsymbol{c}_n^{\,v} \\ &n = 0, 1, 2, \ldots \end{aligned} \tag{15}$$

where ϵ, μ are arbitrary constants and

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{b}_n \rangle &= \int_{\pm\infty}^{x} (\langle \boldsymbol{b}^u_n, \boldsymbol{u}_x \rangle + \langle \boldsymbol{u} \times K(\boldsymbol{v}), \boldsymbol{b}^u_n \rangle + \langle \boldsymbol{v} \times K(\boldsymbol{u}), \boldsymbol{c}_n^{\,v} \rangle)\, \mathrm{d}x \\ \langle \boldsymbol{v}, \boldsymbol{c}_n \rangle &= \int_{\pm\infty}^{x} (\langle \boldsymbol{c}_n^{\,u}, \boldsymbol{v}_x \rangle + \langle \boldsymbol{u} \times K(\boldsymbol{v}), \boldsymbol{b}^u_n \rangle + \langle \boldsymbol{v} \times K(\boldsymbol{u}), \boldsymbol{c}_n^{\,v} \rangle)\, \mathrm{d}x. \end{aligned} \tag{16}$$

In the above formulae we denote by $\langle , \rangle$ the usual $\mathbb{R}^3$ scalar product:

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{17}$$

and by the upper indices 'u' and 'v' are denoted the projections of the corresponding vector fields onto the planes orthogonal to the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively. (Of course, as $\boldsymbol{u}$ and $\boldsymbol{v}$ depend on x these planes also depend on x.)

Let us remark that the above expressions entail the existence of the integro-differential operators $\mathbf{A}_\pm$, such that

$$\begin{pmatrix} \boldsymbol{b}^u_{n+1} \\ \boldsymbol{c}^{\,v}_{n+1} \end{pmatrix} = \mathbf{A}_\pm \begin{pmatrix} \boldsymbol{b}^u_n \\ \boldsymbol{c}_n^{\,v} \end{pmatrix}. \tag{18}$$

The expressions for these operators are as follows

$$\mathbf{A}_{\pm}\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u} \times (\boldsymbol{a})_x + \boldsymbol{u} \times \boldsymbol{u}_x \int_{\pm\infty}^{x} (\langle \boldsymbol{a}, \boldsymbol{u}_x\rangle + \langle \boldsymbol{u} \times K(\boldsymbol{v}), \boldsymbol{a}\rangle + \langle \boldsymbol{v} \times K(\boldsymbol{u}), \boldsymbol{b}\rangle)\, \mathrm{d}x \\ \boldsymbol{v} \times (\boldsymbol{b})_x + \boldsymbol{v} \times \boldsymbol{v}_x \int_{\pm\infty}^{x} (\langle \boldsymbol{b}, \boldsymbol{v}_x\rangle + \langle \boldsymbol{u} \times K(\boldsymbol{v}), \boldsymbol{a}\rangle + \langle \boldsymbol{v} \times K(\boldsymbol{u}), \boldsymbol{b}\rangle)\, \mathrm{d}x \end{pmatrix}$$
$$+ \begin{pmatrix} [K(\boldsymbol{b})]^u - [K(\boldsymbol{v})]^u \int_{\pm\infty}^{x} (\langle \boldsymbol{a}, \boldsymbol{u}_x\rangle - \langle \boldsymbol{b}, \boldsymbol{v}_x\rangle)\, \mathrm{d}x \\ [K(\boldsymbol{a})]^v + [K(\boldsymbol{u})]^v \int_{\pm\infty}^{x} (\langle \boldsymbol{a}, \boldsymbol{u}_x\rangle - \langle \boldsymbol{b}, \boldsymbol{v}_x\rangle)\, \mathrm{d}x \end{pmatrix}$$
$$+ \begin{pmatrix} \boldsymbol{u} \times K(\boldsymbol{v} \times \boldsymbol{b}) + \langle \boldsymbol{u}, K(\boldsymbol{v})\rangle \boldsymbol{a} \\ \boldsymbol{v} \times K(\boldsymbol{u} \times \boldsymbol{a}) + \langle \boldsymbol{u}, K(\boldsymbol{v})\rangle \boldsymbol{b} \end{pmatrix}$$

where $\boldsymbol{a}, \boldsymbol{b}$ are two vector fields, such that $\langle \boldsymbol{a}, \boldsymbol{u}\rangle = \langle \boldsymbol{b}, \boldsymbol{v}\rangle = 0$. We shall see that the operators (18) called recursion operators play a crucial role in the geometrical approach to the theory of the nonlinear evolution equations contained in the CF hierarchy.

3. Two-parametric family of Lie brackets over $so(4)$ and related structures

We intend to show that the Hamiltonian properties of the CF hierarchy of equations are due to the special geometric structures existing over the manifold of potentials. For this we need some preliminaries which we introduce in the next section.

3.1. Two-parametric family of Lie brackets over $so(n)$

Let $so(n)$ $(n > 2)$ be the Lie algebra of skew-symmetric $n \times n$ matrices (the considerations below are the same for both $\mathbb{R}$ and $\mathbb{C}$ so we do not fix the field of numbers). Let J be fixed symmetric matrix $J \in sl(n)$. We can then define the following bilinear skew-symmetric map

$$\begin{aligned} &C : so(n) \times so(n) \to so(n) \\ &C(X, Y) = XJY - YJX. \end{aligned} \tag{19}$$

Clearly, if we fix the first argument in $C(X, Y)$ we obtain a linear map:

$$\begin{aligned} &C_X : so(n) \to so(n) \\ &C_X(Y) \equiv C(X, Y). \end{aligned} \tag{20}$$

Of course the map $X \to C_X$ is also linear and maps $so(n)$ into $\mathrm{Hom}(so(n), so(n))$. One can check by a simple computation that the following proposition holds.

Proposition 3.1. For arbitrary $X, Y \in so(n)$

$$[C_X, C_Y] = C_{C(X,Y)}. \tag{21}$$

Corollary 3.1. The vector space $so(n)$ can be endowed not only with the usual Lie algebra structure defined by the commutator but with an additional Lie algebra structure defined by $C(X, Y)$.

Proof. Indeed, $C(X, Y)$ is bilinear and skew-symmetric and the Jacobi identity is equivalent to (21). □

The map $C(X, Y)$ possesses other interesting properties, which we introduce below.

Proposition 3.2. $C(X, Y)$ is two-coboundary for the adjoint representation of $so(n)$.

Proof. Let us consider the linear map α:

$$\begin{aligned} &\alpha : so(n) \to so(n) \\ &\alpha(X) = \tfrac{1}{2}(JX + XJ) \end{aligned} \tag{22}$$

where $X \in so(n)$ and J is the same diagonal matrix that we used in the definition of $C(X, Y)$. Then it is readily seen that $\mathrm{d}\alpha(X, Y) = C(X, Y)$. As $\mathrm{d}^2 = 0$, for arbitrary $X, Y, Z \in so(n)$ we have $\mathrm{d}C(X, Y, Z) = 0$. $\square$

From the above proposition one readily obtains the following.

Corollary 3.2. For arbitrary $X, Y \in so(n)$

$$[\mathrm{ad}_X, C_Y] + [C_X, \mathrm{ad}_Y] - C_{[X,Y]} - \mathrm{ad}_{C(X,Y)} = 0. \tag{23}$$

We would like to also note the following property of the map α in the case when our algebra is $so(4)$ and the element J is the same as in (9).

Proposition 3.3. The map α corresponding to J, defined in (9), interchanges the two $so(3)$ subalgebras of $so(4)$. More precisely:

$$\alpha(\{\boldsymbol{u}\}_I) = -\{K\boldsymbol{u}\}_{II} \qquad \alpha(\{\boldsymbol{u}\}_{II}) = -\{K\boldsymbol{u}\}_I \tag{24}$$

where $K = \mathrm{diag}(j_1, j_2, j_3)$.

The following theorem can be proved by a simple computation.

Theorem 3.1. Let $C(X, Y)$ be a map for which (23) holds. Then for arbitrary numbers a, b the bilinear map

$$\begin{aligned} &H^{(a,b)} : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \\ &H^{(a,b)}(X, Y) = a[X, Y] + bC(X, Y) \end{aligned} \tag{25}$$

defines a Lie algebra structure over $so(n)$.

Remark 3.1. When $C(X, Y)$ is defined by J, as in (19), the proof of the above statement can be simplified as the bilinear map $H(X, Y)$ is constructed in the same way as the map $C(X, Y)$, but using instead of J the matrix $aJ + b\mathbf{1}_n$.

If $so(n)$ is endowed with the structure defined by the new commutator $H^{(a,b)}(X, Y)$ related to the fixed symmetric matrix J we shall denote it by $so(n)_{(a,b)}$. We shall denote by $H_X^{(a,b)}$ the linear map that corresponds to $H^{(a,b)}(X, Y)$ when we fix the first argument.

3.2. *Invariant bilinear forms for $so(4)_{(a,b)}$*

We shall now concentrate on the algebra $so(4)$. The algebra $so(4)$ is semisimple but not a simple one, ($so(4) = so(3) \oplus so(3)$, see (5)). For a given Lie algebra $\mathcal{G}$ let us introduce the Killing form $B(X, Y)$:

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \qquad X, Y \in \mathcal{G}. \tag{26}$$

$B(X, Y)$ is symmetric and as it is well known it is invariant with respect to the adjoint action, that is:

$$B(\mathrm{ad}_X(Y), Z) + B(Y, \mathrm{ad}_X(Z)) = 0 \qquad X, Y, Z \in \mathcal{G}. \tag{27}$$

An algebra $\mathcal{G}$ (over $\mathbb{R}$ or $\mathbb{C}$) is semisimple if the Killing form $B(X, Y)$ is nondegenerate. If the algebra $\mathcal{G}$ is simple every invariant symmetric bilinear form is proportional to $B(X, Y)$ [16]. On the other hand, if $\mathcal{G}$ is a matrix algebra with respect to the usual commutator, then the form $\mathrm{tr}(XY)$ is symmetric and invariant. It follows that if the trace form is nondegenerate and the algebra is simple then the form $B(X, Y)$ is proportional to $\mathrm{tr}(XY)$. It turns out that in the case $\mathcal{G} = so(4)$ the Killing form $B(X, Y)$ is again proportional to $\mathrm{tr}(XY)$ despite the fact that $so(4)$ is semisimple but not a simple one and we shall use this fact in the calculations as the trace form is simpler. For our considerations it is also important that when $\mathcal{G}$ is semisimple but not simple there are other invariant bilinear forms which are not proportional to the Killing form. For $\mathcal{G} = so(4)$ we shall introduce such a form. In order to do so let us define the linear map $T : so(4) \to so(4)$

$$(T(X))_{ij} = \tfrac{1}{2}\epsilon_{ijks}X_{ks} \qquad i, j, k, s = 1, 2, 3, 4. \tag{28}$$

In the above expression ϵ_{ijks} is the alternating Levi–Civita symbol, that is ϵ_{ijks} is equal to zero if at least two of its indices coincide and if (i, j, k, s) is a permutation of $(1, 2, 3, 4)$ then ϵ_{ijks} is equal to the parity of the permutation (i, j, k, s). We also note that in (28) the usual rule about the summation over repeated indices is assumed. The next proposition can be proved by direct calculation.

Proposition 3.4. Let $\alpha(X)$ be the map $\alpha(X) = \frac{1}{2}(JX + XJ)$ with J defined in (9) and T be the map defined in (28). Then:

(1) T is involutive:

$$T^2 = \mathrm{id}_{so(4)}.$$

(2) T is symmetric with respect to the Killing form:

$$B(T(X), Y) = B(T(Y), X) \qquad X, Y \in so(4).$$

(3) $[T(X), T(Y)] = [X, Y] \qquad X, Y \in so(4)$.

(4) The two $so(3)$ subalgebras of $so(4)$ are invariant under the action of T:

$$T(\{\boldsymbol{u}\}_I) = \{\boldsymbol{u}\}_I$$
$$T(\{\boldsymbol{u}\}_{II}) = -\{\boldsymbol{u}\}_{II}.$$

(5) The form $B([X, Y], T(Z))$ is three-cocycle for the trivial representation of $so(4)$.

(6) The form $B(XJY - YJX, T(Z)) = B(C(X, Y), T(Z))$ is three-cocycle for the trivial representation of $so(4)$.

(7) The linear mapping α is symmetric with respect to the Killing form:

$$B(\alpha(X), Y) = B(X, \alpha(Y)) \qquad X, Y \in so(4).$$

(8) $T \circ \alpha = -\alpha \circ T$.

Let us define the bilinear form

$$K(X, Y) \equiv B(X, T(Y)) \qquad X, Y \in so(4). \tag{29}$$

Taking into account the properties of T it is not hard to prove the following.

Proposition 3.5. $K(X, Y)$ is invariant bilinear form with respect to the adjoint action of $so(4)_{(a,b)}$.

From the properties of T and α listed in proposition 3.4 we get the following.

Proposition 3.6. The linear mapping α is skew-symmetric with respect to the form K, that is

$$K(\alpha(X), Y) + K(X, \alpha(Y)) = 0 \qquad X, Y \in so(4). \tag{30}$$

3.3. Two-parametric family of Poisson–Lie tensors over $so(4)$

As is well known (see for example [25]) the Poisson brackets over a manifold† $\mathcal{M}$ can be introduced either by symplectic form or by Poisson tensor. Let us recall some principal results from the theory of Poisson manifolds, that is the manifolds equipped with Poisson tensor.

Let $T_m(\mathcal{M})$ and $T_m^*(\mathcal{M})$ be the tangent and the cotangent spaces at the point m of the manifold $\mathcal{M}$.

The Poisson tensor field (or simply Poisson tensor) is a field of linear mappings $m \to P_m : T_m^*(\mathcal{M}) \to T_m(\mathcal{M})$ having the properties:

$$\text{(i) } P_m^* = -P_m \qquad \text{(ii) } [P, P]_S = 0.$$

Here $[\,.\,]_S$ denotes the so-called Schouten bracket of two tensor fields (see [9]).

The Poisson tensors were discovered by Lie [20] and have many applications, see for example [19, 25]. Condition (ii) ensures the Jacobi identity and (i) guarantees the skew-symmetry of the Poisson bracket

$$\{f, g\} = (P\mathrm{d}f, \mathrm{d}g)$$

where f and g are functions over the manifold $\mathcal{M}$.

Remark 3.2. When we write $P^* = -P$ we assume that the spaces $T_m^{**}(\mathcal{M})$ and $T_m(\mathcal{M})$ can be identified. This of course is always possible if the manifold $\mathcal{M}$ is finite dimensional, but if it is not the case one must proceed with some caution.

The Poisson bracket could be degenerate, that is P does not necessarily possess an inverse. When P is invertible one can define the symplectic form ω through the formula $\omega(X, Y) = (X, P^{-1}(Y))$, where X, Y are vector fields over $\mathcal{M}$. Naturally, in this case the Poisson brackets defined through the Poisson structure and through symplectic structure ω coincide.

There is a canonical way to equip the dual space $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$ with Poisson structure, provided one can identify the vector spaces $\mathcal{G}^{**}$ and $\mathcal{G}$. This structure is again discovered by Lie, [20] and was rediscovered later by several authors, see for example [17, 34].

Suppose $\mu \in \mathcal{G}^*$. Then clearly

$$T_\mu(\mathcal{G}^*) = \mathcal{G}^* \qquad T_\mu^*(\mathcal{G}^*) = \mathcal{G}^{**} = \mathcal{G}. \tag{31}$$

The canonical Poisson structure over $\mathcal{G}^*$ is defined by the following field of linear maps:

$$\begin{aligned} &\mu \to L_\mu \in \mathrm{Hom}(\mathcal{G}, \mathcal{G}^*) \\ &L_\mu(\xi) = -\,\mathrm{ad}_\xi^*\,\mu \qquad \xi \in \mathcal{G}. \end{aligned} \tag{32}$$

The tensor L is called Poisson–Lie tensor or Kirillov tensor and the Poisson bracket defined by it is called Poisson–Lie bracket, Kirillov bracket, Berezin bracket etc. We shall call the above tensor the Poisson–Lie tensor and the corresponding bracket the Poisson–Lie bracket.

If there exists symmetric nondegenerate bilinear $B(X, Y)$ over $\mathcal{G}$,invariant with respect to the adjoint action of $\mathcal{G}$, then one can not only identify in canonical way $\mathcal{G}^*$ and $\mathcal{G}$ but define Poisson brackets on $\mathcal{G}$. Indeed, if $\mathcal{G}^*$ and $\mathcal{G}$ are identified the adjoint and the coadjoint representations coincide and one can define Poisson structure over the algebra $\mathcal{G}$:

$$\begin{aligned} &q \to L_q \in \mathrm{Hom}(\mathcal{G}, \mathcal{G}) \\ &L_q(\xi) = \mathrm{ad}_\xi\, q \qquad \xi \in \mathcal{G} \equiv \mathcal{G}^*. \end{aligned} \tag{33}$$

† In what follows we assume that all the manifolds and all the tensor fields are smooth.

If f, g are smooth functions over $\mathcal{G}$ the Poisson bracket $\{f, g\}$ of these functions is then

$$\{f, g\}(q) = -\langle q, [\mathrm{d}f_q, \mathrm{d}g_q]\rangle = -B(q, [\mathrm{d}f_q, \mathrm{d}g_q]) \tag{34}$$

where the differentials $\mathrm{d}f_q, \mathrm{d}g_q \in \mathcal{G}^* \equiv \mathcal{G}$.

Considering the algebraical structure over $so(4)_{(a,b)}$ and taking into account the existence of the invariant bilinear form $K(X, Y)$ we arrive at the following theorem.

Theorem 3.2. For fixed element J there exists a two-parametric family of Poisson–Lie structures over the manifold $\mathcal{M} = so(4)$:

$$\begin{aligned} &q \to P_q^{(a,b)} \in \mathrm{Hom}(so(4), so(4)) \\ &P_q^{(a,b)}(\xi) = H_\xi^{(a,b)}(q) = -H_q^{(a,b)}(\xi) \qquad \xi \in so(4) \equiv so(4)^*. \end{aligned} \tag{35}$$

3.4. The Gel'fand–Fuchs cocycle and related Poisson structures

There is an elegant way to define the Poisson tensor over the infinite-dimensional manifold $\mathcal{G}_0[x]$ of all smooth functions $f(x)$ defined on the real line, taking their values in the Lie algebra $\mathcal{G}$ and tending fast enough to constant $f_0 \in \mathcal{G}$ when $|x| \to \infty$. For obvious reasons we take the tangent space $T_f(\mathcal{G}_0[x])$ at the point $f \in \mathcal{G}_0[x]$ to be the vector space consisting of all Schwartz-type functions $\xi(x)$ on the line taking their values in $\mathcal{G}$. We shall denote this space by $\mathcal{G}[x]$. If we define the Lie algebra operation pointwise both $\mathcal{G}_0[x]$ and $\mathcal{G}[x]$ become Lie algebras and

$$[\mathcal{G}[x], \mathcal{G}_0[x]] \subset \mathcal{G}[x]. \tag{36}$$

First, we recall that if $\mathcal{G}$ is a Lie algebra and γ is a two-cocycle of the trivial action of the algebra on the field of scalars, then

$$\mu \to P_\mu : P_\mu(\xi) = -\,\mathrm{ad}_\xi^*\, \mu + \gamma(\xi, .) \qquad \xi \in \mathcal{H} \tag{37}$$

is Poisson tensor over $\mathcal{G}^*$.

The above trick is widely applied in the theory of the integrable equations in the case $\mathcal{H} = \mathcal{G}[x]$, the algebra of Schwartz-type functions on $\mathbb{R}$ taking values in $\mathcal{G}$, see for example [28–31, 25]. If $K(X, Y)$ is invariant nondegenerate bilinear form on $\mathcal{G}$ then we can define the following two-cocycle of $\mathcal{G}[x]$, called the Gel'fand–Fuchs cocycle:

$$\gamma(\xi, \eta) = c \int_{-\infty}^{+\infty} K(\partial_x \xi, \eta(x))\, \mathrm{d}x \qquad \xi, \eta \in \mathcal{H}[x] \qquad c = \text{constant} \qquad \partial_x \equiv \frac{\partial}{\partial x}. \tag{38}$$

Allowing some lack of rigor we identify $\mathcal{G}[x]$ and $\mathcal{G}[x]^*$ using the invariant nondegenerate bilinear form on $\mathcal{G}[x]$:

$$\langle\langle \xi, \eta \rangle\rangle = \int_{-\infty}^{+\infty} K(\xi(x), \eta(x))\, \mathrm{d}x \qquad \xi, \eta \in \mathcal{G}[x] \tag{39}$$

and as a result we obtain the following Poisson tensor

$$P_q(\xi) = [\xi, q] + c\partial_x \xi. \tag{40}$$

It can be verified that the above expression actually defines the Poisson tensor also for the case when the potential q belongs to the set of all smooth functions with values in $\mathcal{G}$ tending fast enough to constant element from $\mathcal{G}$ as $|x| \to \infty$, that is to $\mathcal{G}_0[x]$. Applying the above constructions for the case $\mathcal{G} = so(4)_{(a,b)}$ we obtain the following theorem.

Theorem 3.3. Over the manifold $so(4)_0[x]$ there exists a three-parametric family of Poisson tensor fields:

$$P_q^{(a,b,c)} = -H_q^{(a,b)} + c\partial_x. \tag{41}$$

The situation stated in the above theorem is quite similar to that described in [31] but the existence of scalar product invariant with respect to the both Lie algebra structures makes the $so(4)$ case quite unique.

4. Geometric setting for the CF hierarchy

4.1. Poisson–Nijenhuis structures.

In order to develop the geometric theory for the CF hierarchy we recall some properties of the so-called Poisson–Nijenhuis (P–N) manifolds. It is known that they give geometric interpretations for some of the remarkable properties of the soliton equations.

Let $\mathcal{M}$ be a manifold. A Nijenhuis tensor field N over $\mathcal{M}$ (or simply a Nijenhuis tensor) is a field of continuous linear mappings:

$$m \to N_m : T_m(\mathcal{M}) \to T_m(\mathcal{M}) \qquad m \in \mathcal{M} \tag{42}$$

(field of operators) for which the Nijenhuis bracket $R \equiv [N, N]$ vanishes, see [9]. Roughly speaking the fact that $[N, N] = 0$ ensures that the eigenspaces of N are integrable in Frobenius sense. (For some additional properties of the Nijenhuis tensors see [35, 36].)

Following [23] we shall say that on the manifold $\mathcal{M}$ is defined a P–N structure if on $\mathcal{M}$ are defined simultaneously Poisson tensor P and Nijenhuis tensor N which satisfy the following coupling conditions:

$$NP = PN^* \tag{43a}$$

$$PL_{N(X)}(\alpha) - PL_X(N^*\alpha) + L_{P\alpha}(N)(X) = 0 \tag{43b}$$

for an arbitrary choice of $X \in \mathcal{T}(\mathcal{M})$ and $\alpha \in \mathcal{T}^*(\mathcal{M})$. (Here by $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$ are denoted the modules of vector fields and one-forms over $\mathcal{M}$ and L_X means the Lie derivative defined by the vector field X.) The above structure seems very specific, but for the soliton equations it is natural. In fact in almost every approach to the theory of completely integrable systems one can notice that a crucial role is played by the so-called compatible Poisson tensors, see [5, 6, 21, 22], or as they are also called Hamiltonian pairs, see [11, 12]. Two Poisson tensors P and Q are compatible if the tensor $P + Q$ is also a Poisson tensor. It is evident that for this it is necessary and sufficient that

$$[P, Q]_S = 0 \tag{44}$$

where $[P, Q]_S$ is the Schouten bracket. It can be shown [23], that the compatible Poisson tensors define the P–N structure, more specifically if P and Q are Poisson tensors on the manifold $\mathcal{M}$ and Q is invertible then the tensor fields $N = P \circ Q^{-1}$ and Q endow the manifold $\mathcal{M}$ with P–N structure. The properties of the P–N manifolds (or compatible Poisson tensors) explain some of the remarkable features of the soliton equations, such as: the fact that they appear in hierarchies, the existence of series of conservation laws for these equations and the fact that they are Hamiltonian with respect to hierarchies of Poisson structures (symplectic structures), see [11, 12, 21, 22]. According to the geometric schemes the soliton equations are defined by the fundamental fields of the above structures, for example in the case of P–N manifolds the fundamental fields X satisfy $L_X(P) = 0$, $L_X(N) = 0$.

Naturally, it happens that Poisson tensor Q is degenerate, that is Q^{-1} do not exist and it is impossible to perform the construction outlined in the above. However, if it is possible to restrict the Poisson tensors P and Q onto some submanifold $\mathcal{N} \subset \mathcal{M}$ and if the restriction of Q is nondegenerate then $\mathcal{N}$ is endowed with the P–N structure. That is why it is important to know how to restrict Poisson tensors onto submanifolds. One of the results treating this issue is the following theorem, [23, 24], which we shall refer to as the *restriction theorem*.

Theorem 4.1. Let $\mathcal{M}$ be a Poisson manifold and $\mathcal{N} \subset \mathcal{M}$ be a submanifold. Let us denote by i the natural inclusion of $\mathcal{N}$ into $\mathcal{M}$, by $\chi^*_P(\mathcal{N})_m$ the subspace of covectors $\alpha \in T^*_m(\mathcal{M})$ such that

$$P_m(\alpha) \in [di]_m(T_m(\mathcal{N})) = im([di]_m) \qquad m \in \mathcal{N} \tag{45}$$

and by $T^{\perp}_m(\mathcal{N})$ the set of all covectors over $\mathcal{M}$ vanishing on the subspace $im([di]_m)$, $m \in \mathcal{N}$ (the annihilator of $im([di]_m)$ in $T^*_m(\mathcal{M})$). Let the following relations hold:

$$\chi^*_P(\mathcal{N})_m + T^{\perp}_m(\mathcal{N}) = T^*_m(\mathcal{M}) \qquad m \in \mathcal{N} \tag{46}$$

$$\chi^*_P(\mathcal{N})_m \cap T^{\perp}_m(\mathcal{N}) \subset \ker(P_m) \qquad m \in \mathcal{N}. \tag{47}$$

Then there exists a unique Poisson tensor $\overline{P}$ on $\mathcal{N}$, i-connected with P, that is

$$P_m = [di]_m \circ \overline{P}_m \circ [di]^*_m \qquad m \in \mathcal{N}. \tag{48}$$

The above theorem may be applied for example to restrict the Poisson–Lie tensor L_μ for a given algebra $\mathcal{G}$ onto the orbits of the coadjoint representation of the Lie group G, corresponding to $\mathcal{G}$, see [17, 34] and for general results about restriction of Poisson tensors see [26].

4.2. *P–N structure of the manifold of potentials for the CF hierarchy*

We shall define now the Hamiltonian structures of the equations from the CF hierarchy. First, let us remark that the set of potentials has a natural structure of infinite-dimensional manifold. Indeed, the manifold of potentials for the CF hierarchy is in fact the set of smooth functions

$$\begin{aligned} &A(x) = \{\boldsymbol{u}(x)\}_I + \{\boldsymbol{v}(x)\}_{II} \\ &(\boldsymbol{u}(x))^2 = (\boldsymbol{v}(x))^2 = 1 \\ &\boldsymbol{u}(x), \boldsymbol{v}(x) - \text{real} \end{aligned} \tag{49}$$

defined over the real line $\mathbb{R}$ and tending fast enough to some limit value

$$A_0 = \{\boldsymbol{u}_0\}_I + \{\boldsymbol{v}_0\}_{II}$$

as $|x| \to \infty$.

Diagonalizing the matrix $A(x)$ we see that the requirements (49) simply mean that $A(x)$ takes its values into the following orbit of the adjoint representation of the group $SO(4, \mathbb{R})$:

$$\mathcal{O}_{B_0} = \{A = Ad(g)B_0 \qquad g \in SO(4, \mathbb{R})\} \subset so(4) \tag{50}$$

where

$$B_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \{(1, 0, 0)\}_I + \{(1, 0, 0)\}_{II}. \tag{51}$$

Therefore the manifold of potentials is the set $\mathcal{O}_{B_0}[x]$ consisting of functions taking values in $\mathcal{O}_{B_0}$ and tending fast enough to some limit value as $|x| \to \infty$. Clearly $\mathcal{O}_{B_0}[x]$ is an infinite-dimensional manifold and submanifold of $so(4, \mathbb{R})_0[x]$. This submanifold can be expressed more explicitly if one remarks that the orbit $\mathcal{O}_{B_0}$ has also the following representation:

$$\mathcal{O}_{B_0} = \{A : B(A, A) = -16 \qquad K(A, A) = 0\} \subset so(4, \mathbb{R}) \tag{52}$$

where $B(X, Y)$ and $K(X, Y)$ are the symmetric forms on $so(4, \mathbb{R})$ that we introduced earlier. In order to see that it is enough to remark that for arbitrary $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$

$$\begin{aligned} &B(\boldsymbol{a}_I, \boldsymbol{b}_I) = K(\boldsymbol{a}_I, \boldsymbol{b}_I) = -8\langle \boldsymbol{a}, \boldsymbol{b}\rangle \\ &B(\boldsymbol{a}_{II}, \boldsymbol{b}_{II}) = -K(\boldsymbol{a}_{II}, \boldsymbol{b}_{II}) = -8\langle \boldsymbol{a}, \boldsymbol{b}\rangle \\ &B(\boldsymbol{a}_I, \boldsymbol{b}_{II}) = K(\boldsymbol{a}_I, \boldsymbol{b}_{II}) = 0. \end{aligned} \tag{53}$$

Naturally $X(x) \in T_A(\mathcal{O}_{B_0}[x])$ exactly when

$$B(A, X) = 0 \qquad K(A, X) = B(A, T(X)) = B(T(A), X) = 0. \tag{54}$$

For the sake of brevity we shall denote the points $Q = \{\boldsymbol{a}\}_I + \{\boldsymbol{b}\}_{II}$ of the algebra $so(4)$ by $Q = (\boldsymbol{a}, \boldsymbol{b})^T$ and the points $A(x) = \{\boldsymbol{u}(x)\}_I + \{\boldsymbol{v}(x)\}_{II}$ of the manifold $\mathcal{O}_{B_0}[x]$ by $A(x) = (\boldsymbol{u}(x), \boldsymbol{v}(x))^T$ or simply by $A = (\boldsymbol{u}, \boldsymbol{v})^T$. Also, in order to write in a more convenient way some complicated expressions we shall denote by lower indices I and II the following projections

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}_I = \boldsymbol{\xi} \qquad \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}_{II} = \boldsymbol{\eta}. \tag{55}$$

With the new notations a vector $X(x)$ at the point $A(x) \in \mathcal{O}_{B_0}[x]$ is represented by a couple of Schwartz-type functions $(\boldsymbol{\xi}(x), \boldsymbol{\eta}(x))^T$, for which:

$$\langle \boldsymbol{u}(x), \boldsymbol{\xi}(x)\rangle = \langle \boldsymbol{v}(x), \boldsymbol{\eta}(x)\rangle = 0. \tag{56}$$

According to our convention we identify the vectors and covectors using the pairing defined in (39). We easily obtain

$$\langle\langle (\boldsymbol{\xi}(x), \boldsymbol{\eta}(x))^T, (\boldsymbol{\mu}(x), \boldsymbol{\nu}(x))^T\rangle\rangle = -8\int_{-\infty}^{+\infty} [\langle \boldsymbol{\xi}(x), \boldsymbol{\mu}(x)\rangle - \langle \boldsymbol{\eta}(x), \boldsymbol{\nu}(x)\rangle]\, \mathrm{d}x. \tag{57}$$

Thus $K(X, Y)$ is nondegenerate when restricted to the tangent space $T_A(\mathcal{O}_{B_0}[x])$ and as before with the help of $K(X, Y)$ we can identify the tangent space $T_A(\mathcal{O}_{B_0}[x])$ and the cotangent space $T_A^*(\mathcal{O}_{B_0}[x])$.

We shall now try to restrict two of the tensors from the three-parametric family of Poisson tensors defined in theorem 3.3 onto the submanifold $\mathcal{O}_{B_0}[x]$. These tensors are

$$\begin{aligned} &Q_A \equiv -H_A^{(-\frac{1}{2},0)} = \tfrac{1}{2}\,\mathrm{ad}_A \\ &P_A \equiv -H_A^{(0,\frac{1}{2})} + \partial_x = -\tfrac{1}{2} C_A + \partial_x \end{aligned} \tag{58}$$

or equivalently

$$P\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\xi}_x + K(\boldsymbol{v} \times \boldsymbol{\eta}) - \boldsymbol{u} \times K(\boldsymbol{\eta}) + \boldsymbol{\xi} \times K(\boldsymbol{v}) \\ \boldsymbol{\eta}_x + K(\boldsymbol{u} \times \boldsymbol{\xi}) - \boldsymbol{v} \times K(\boldsymbol{\xi}) + \boldsymbol{\eta} \times K(\boldsymbol{v}) \end{pmatrix} \tag{59}$$

$$Q\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{u} \times \boldsymbol{\xi} \\ -\boldsymbol{v} \times \boldsymbol{\eta} \end{pmatrix}. \tag{60}$$

Remark that $Q^{-1} = -Q$.

The fact that Q allows nondegenerate restriction over $\mathcal{O}_{B_0}[x]$ and its form after the restriction does not change is in fact the theorem that the Poisson–Lie tensor restricted to an orbit of the coadjoint representation is nondegenerate, so there is no need to prove it.

In view of the tensor P it cannot be restricted directly. In order to perform the restriction we shall use theorem 4.1 with $\mathcal{M} = so(4, \mathbb{R})_0[x]$ and $\mathcal{N} = \mathcal{O}_{B_0}[x]$. Let us find $\chi_P^*(\mathcal{N})_A$ and $T_A^\perp(\mathcal{N})$, $A \in \mathcal{N} = \mathcal{O}_{B_0}[x]$ (for the definitions of these spaces see the restriction theorem 4.1). Naturally, the annihilator

$$T_A^\perp(\mathcal{N}) = \{(f\boldsymbol{u}, g\boldsymbol{v})^T;\ \ f, g \in \mathcal{S}\} \tag{61}$$

where $\mathcal{S}$ is the set of all Schwartz-type functions on the line. We can also say that

$$T_A^\perp(\mathcal{N}) = \{(\bar{f}A + \bar{g}T(A);\ \ \bar{f}, \bar{g} \in \mathcal{S}\}. \tag{62}$$

According to the definition $X \in \chi_P^*(\mathcal{N})_A$ if $P_A(X) \in T_A(\mathcal{O}_{B_0}[x])$ or in other words if the following equations hold

$$B(A, \partial_x X) - \tfrac{1}{2}B(A, C_A(X)) = 0 \tag{63}$$
$$K(A, \partial_x X) = 0. \tag{64}$$

After some simple transformations we obtain that these equations are equivalent to

$$B(A, X) = \partial_x^{-1}[B(A_x, X) + \tfrac{1}{2}B(A, C_A(X))] \tag{65}$$
$$K(A, X) = B(T(A), X) = \partial_x^{-1}B(A_x, T(X)) \tag{66}$$

where ∂_x^{-1} stands for the inverse of the operator ∂_x. The choice of ∂_x^{-1} of course is not unique and it is easy to see that we can use as inverse any of the operators:

$$\partial_x^{-1} = \tau\int_{-\infty}^{x} + (1-\tau)\int_{+\infty}^{x} \qquad \tau \in \mathbb{R} \tag{67}$$

but we shall postpone the discussion about the appropriate choice for ∂_x^{-1} in order to proceed with our geometric construction.

Let us remark that for $A \in \mathcal{O}_{B_0}[x]$ we have $B(A, T(A)) = K(A, A) = 0$ or in other words A and $T(A)$ are orthogonal with respect to the Killing form. Then taking into account (54) we see that the following orthogonal decomposition holds:

$$so(4, \mathbb{R}) = [\mathbb{R}A(x) \oplus \mathbb{R}T(A(x))] \oplus T_A(\mathcal{O}_{B_0}[x]). \tag{68}$$

(This decomposition obviously depends on x.)

For fixed X let us denote by X^A the orthogonal projection of X onto the space $T_A(\mathcal{O}_{B_0}[x])$ and the orthogonal projection of X over the space spanned by A and $T(A)$ by X_A. It is easily seen that

$$\begin{aligned} X^A &= X + \tfrac{1}{16}B(A, X)A + \tfrac{1}{16}B(T(A), X)T(A) \\ X_A &= -\tfrac{1}{16}B(A, X)A - \tfrac{1}{16}B(T(A), X)T(A). \end{aligned} \tag{69}$$

If we now return to equations (65) and (66) then due to the fact that

$$B(A, C_A(T(A))) = K(T(A), C_A(T(A))) = 0 \tag{70}$$

we see that in the right-hand sides we can put instead of X the projection X^A and (65), (66) actually show that if $X \in \chi_P^*(\mathcal{N})_A$ the component X_A is expressed by the component X^A. Taking this into account we write

$$X = Y + Z \tag{71}$$

where

$$Y = X^A - \tfrac{1}{16} A \partial_x^{-1}[B(A_x, X^A) + \tfrac{1}{2} B(A, C_A(X^A))] - \tfrac{1}{16} T(A) \partial_x^{-1} B(A_x, T(X^A)) \tag{72}$$

$$Z = X_A + \tfrac{1}{16} A \partial_x^{-1}[B(A_x, X^A) + \tfrac{1}{2} B(A, C_A(X^A))] + \tfrac{1}{16} T(A)\ \partial_x^{-1} B(A_x, T(X^A)). \tag{73}$$

Using (70) one can check that $Y \in \chi_P^*(\mathcal{N})_A$. In view of the vector Z it is a linear combination of A and $T(A)$ and hence $Z \in T_A^{\perp}(\mathcal{N})$. Moreover, from (71) and (65), (66) we see that

$$T_A^{\perp}(\mathcal{N}) \oplus \chi_P^*(\mathcal{N})_A = T_A^*(\mathcal{M}) \tag{74}$$

$$T_A^{\perp}(\mathcal{N}) \bigcap \chi_P^*(\mathcal{N})_A = \{0\} \subset \ker(P_A). \tag{75}$$

(Of course, here $\mathcal{M} = so(4, \mathbb{R})_0[x]$ and $\mathcal{N} = \mathcal{O}_{B_0}[x]$.) Then the requirements of the restriction theorem are fulfilled and there exists restriction $\bar{P}$ of P defined over $\mathcal{N}$. According to the prescriptions of this theorem for $\alpha \in T_A^*(\mathcal{N})$ we must take $\beta = i^*\alpha$, then represent β as sum $\beta_1 + \beta_2$ in such a way that $\beta_1 \in \chi_P^*(\mathcal{N})_A$ and $\beta_2 \in T_A^{\perp}(\mathcal{N})$ and finally put $\bar{P}(\alpha) \equiv P(\beta_1)$. Here as usual i is the natural inclusion map

$$i : \mathcal{N} \to \mathcal{M}. \tag{76}$$

However, in our case $T_A(\mathcal{M})$ and $T_A^*(\mathcal{M})$ are identified and the pull-back of the inclusion map i is simply the orthogonal projection $X \to X^A$. As it is readily seen the role of the component β_1 here is played by expression (72) where we must put X instead of X^A in the integrands. Finally, we arrive to the following expression for the restricted Poisson tensor:

$$\begin{aligned}\bar{P}_A(X) = \partial_x X &- \tfrac{1}{16} A_x \partial_x^{-1}[B(A_x, X) + \tfrac{1}{2} B(A, C_A(X))] \\ &- \tfrac{1}{16} T(A_x) \partial_x^{-1} B(A_x, T(X)) - \tfrac{1}{16} A[B(A_x, X) + \tfrac{1}{2} B(A, C_A(X))] \\ &- \tfrac{1}{16} T(A) B(A_x, T(X)) - \tfrac{1}{2} C_A(X)\end{aligned} \tag{77}$$

$X \in T_A^*(\mathcal{N}) \sim T_A(\mathcal{N})$.

Remark 4.1. The function A_x tends to zero as $|x| \to \infty$ and $X(x)$ is a function of the Schwartz type, so the integrals in (77) exist. The same is true for the integrals in the expressions for Y and Z, see (72) and (73).

We must also ensure that $\bar{P}$ is skew-symmetric at least in a weak sense, that is we must have

$$\langle\langle \bar{P}_A(X), Y \rangle\rangle = -\langle\langle X, \bar{P}_A(Y) \rangle\rangle \tag{78}$$

for $X, Y \in T_A(\mathcal{O}_{B_0}[x])$. A simple integration by parts shows that for this we must take

$$\partial_x^{-1} = \tfrac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x} \right). \tag{79}$$

The construction of Nijenhuis tensor N is now an easy task. We must calculate $N = \bar{P} Q^{-1} = -\bar{P} Q$ or $N^* = Q^{-1} \bar{P} = -Q \bar{P}$. We obtain:

$$\begin{aligned}N_A^*(X) = -\tfrac{1}{2}[A, \partial_x X] &+ \tfrac{1}{32}[A, A_x] \partial_x^{-1}[B(A_x, X) + \tfrac{1}{2} B(A, C_A(X))] \\ &+ \tfrac{1}{32}[A, T(A_x)] \partial_x^{-1} B(A_x, T(X)) \\ &+ \tfrac{1}{32}[A, T(A)] B(A_x, T(X)) + \tfrac{1}{4}[A, C_A(X)]\end{aligned} \tag{80}$$

$X \in T_A^*(\mathcal{N}) \sim T_A(\mathcal{N})$.

Now let us formulate our main result.

Theorem 4.2. The fields of operators $A \to Q_A$ and $A \to \bar{P}_A$ endow the manifold of potentials $\mathcal{N} = \mathcal{O}_{B_0}[x]$ for the CF hierarchy with P–N structure.

We shall now apply now this result to the CF hierarchy, but first we must write the operators which we have obtained in terms of $\boldsymbol{u}, \boldsymbol{v}$. If we put $X = (\boldsymbol{\xi}, \boldsymbol{\eta})^T$ and assume $\langle \boldsymbol{u}, \boldsymbol{\xi}\rangle = \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle = 0$ we get:

$$\begin{aligned}
\left[\bar{P}_A\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\right]_I &= [\partial_x\boldsymbol{\xi} + K(\boldsymbol{v}\times\boldsymbol{\eta}) + \boldsymbol{u}\times K(\boldsymbol{\eta}) + \boldsymbol{\xi}\times K(\boldsymbol{v})]^u \\
&\quad + \boldsymbol{u}\times K(\boldsymbol{v})\ \partial_x^{-1}[\langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle - \langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle] \\
&\quad + \boldsymbol{u}_x\ \partial_x^{-1}[\langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle + \langle \boldsymbol{u}\times K(\boldsymbol{v}), \boldsymbol{\xi}\rangle - \langle K(\boldsymbol{u})\times\boldsymbol{v}, \boldsymbol{\eta}\rangle]
\end{aligned}\tag{81}$$

$$\begin{aligned}
\left[\bar{P}_A\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\right]_{II} &= [\partial_x\boldsymbol{\eta} + K(\boldsymbol{u}\times\boldsymbol{\xi}) + \boldsymbol{v}\times K(\boldsymbol{\xi}) + \boldsymbol{\eta}\times K(\boldsymbol{u})]^v \\
&\quad + \boldsymbol{v}\times K(\boldsymbol{u})\ \partial_x^{-1}[\langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle - \langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle] \\
&\quad + \boldsymbol{v}_x\ \partial_x^{-1}[\langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle + \langle \boldsymbol{v}\times K(\boldsymbol{u}), \boldsymbol{\eta}\rangle - \langle K(\boldsymbol{v})\times\boldsymbol{u}, \boldsymbol{\xi}\rangle]
\end{aligned}\tag{82}$$

where as before by upper indices u, v we denote the projections over the planes orthogonal to the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively and $K = \mathrm{diag}(j_1, j_2, j_3)$. For the tensor field N^* we obtain:

$$\begin{aligned}
\left[N^*_A\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\right]_I &= \boldsymbol{u}\times\partial_x\boldsymbol{\xi} + \boldsymbol{u}\times K(\boldsymbol{v}\times\boldsymbol{\eta}) - [K(\boldsymbol{\eta})]^u + \boldsymbol{\xi}\langle K(\boldsymbol{v}), \boldsymbol{u}\rangle \\
&\quad - [K(\boldsymbol{v})]^u\ \partial_x^{-1}[\langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle - \langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle] \\
&\quad + \boldsymbol{u}\times\boldsymbol{u}_x\partial_x^{-1}[\langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle + \langle \boldsymbol{u}\times K(\boldsymbol{v}), \boldsymbol{\xi}\rangle - \langle K(\boldsymbol{u})\times\boldsymbol{v}, \boldsymbol{\eta}\rangle]
\end{aligned}\tag{83}$$

$$\begin{aligned}
\left[N^*_A\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\right]_{II} &= \boldsymbol{v}\times\partial_x\boldsymbol{\eta} + \boldsymbol{v}\times K(\boldsymbol{u}\times\boldsymbol{\xi}) - [K(\boldsymbol{\xi})]^v + \boldsymbol{\eta}\langle K(\boldsymbol{u}), \boldsymbol{v}\rangle \\
&\quad - [K(\boldsymbol{u})]^v\partial_x^{-1}[\langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle - \langle \boldsymbol{u}_x, \boldsymbol{\xi}\rangle] \\
&\quad + \boldsymbol{v}\times\boldsymbol{v}_x\ \partial_x^{-1}[\langle \boldsymbol{v}_x, \boldsymbol{\eta}\rangle + \langle \boldsymbol{v}\times K(\boldsymbol{u}), \boldsymbol{\eta}\rangle - \langle K(\boldsymbol{v})\times\boldsymbol{u}, \boldsymbol{\xi}\rangle].
\end{aligned}\tag{84}$$

The comparison shows that the recursion operators from (18) are related to N^* in the following way:

$$N^* = \tfrac{1}{2}(\mathbf{A}_+ + \mathbf{A}_-).\tag{85}$$

For the equations from the CF hierarchy one can equivalently use $\mathbf{A}_+$ and $\mathbf{A}_-$ (the integrands in this formulae are always total derivatives) and therefore it is evident that one can also use N^*. Remember now that the equations from the CF hierarchy have the form (cf (12) and (13)),

$$A_t = (B_n)_x - \tfrac{1}{2}(AJB_n - B_nJA) = \tfrac{1}{2}[A, B_{n+1}] = Q_A(B_{n+1})\tag{86}$$

and

$$\begin{aligned}
&B_{n+1} = N^*(B_n) \qquad n \geqslant 1 \\
&B_n = \{\boldsymbol{b}_n\}_I + \{\boldsymbol{c}_n\}_{II}.
\end{aligned}\tag{87}$$

If we consider B_n as one-forms all of these equations are Hamiltonian. The single thing that must be proved in order to apply the general results about P–N manifolds to our case is to show that the forms B_1 and B_2 are closed. However, the evolution equation corresponding to B_2 is up to some changes of the parameters the O(3) CF system. It is well known that it has a Hamiltonian function, see [33] and therefore B_2 is closed. In view of the form B_1 it is proportional to

$$\epsilon\{\boldsymbol{u}\times\boldsymbol{u}_x\}_I + \mu\{\boldsymbol{v}\times\boldsymbol{v}_x\}_{II}\tag{88}$$

ϵ, μ being constants. One readily sees that it is enough to show that on the manifold $\mathcal{N}$ of the smooth vector functions $\boldsymbol{v}(x)$ taking values on the unit sphere and tending sufficiently fast to some value $\boldsymbol{v}_0$ as $|x| \to \infty$ the covector field $\boldsymbol{v} \to \gamma_v$:

$$\gamma_v(\boldsymbol{\xi}) = \int_{-\infty}^{+\infty} \langle \boldsymbol{v} \times \boldsymbol{v}_x, \boldsymbol{\xi}[v] \rangle \mathrm{d}x \qquad \boldsymbol{\xi}[v] \in T_v(\mathcal{N}) \tag{89}$$

is closed. If $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two vector fields on $\mathcal{N}$ then the calculation shows that

$$[d\gamma]_v(\xi, \eta) = \int_{-\infty}^{+\infty} \partial_x \langle \boldsymbol{v}, \boldsymbol{\xi}[v] \times \boldsymbol{\eta}[v] \rangle \, \mathrm{d}x - 3 \int_{-\infty}^{+\infty} \langle \boldsymbol{v}_x, \boldsymbol{\xi}[v] \times \boldsymbol{\eta}[v] \rangle \, \mathrm{d}x.$$

For the vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ $\lim_{|x|\to 0} \boldsymbol{\xi}[v(x)] = \lim_{|x|\to 0} \boldsymbol{\eta}[v(x)] = 0$ and the first term in the right-hand side is zero. In view of the second term it is zero simply because $\boldsymbol{v}_x(x), \boldsymbol{\xi}[v(x)], \boldsymbol{\eta}[v(x)]$ are orthogonal to $\boldsymbol{v}(x)$ and hence $\langle \boldsymbol{v}_x, \boldsymbol{\xi}[v] \times \boldsymbol{\eta}[v] \rangle = 0$. As a consequence from the above considerations and from the properties of the fundamental fields of the P–N manifolds, see [21, 22], we get the following.

Theorem 4.3. The right hand sides of the equations from the CF hierarchy are fundamental fields for the P–N structure generated by the field $Q^{-1}B_1$. These equations are Hamiltonian with respect to the infinite hierarchy of Poisson structures and the flows corresponding to these fields commute.

5. Discussion

As we have stated, the main question that we are trying to answer is whether using essentially different Lax pairs we obtain the same results for the corresponding nonlinear evolution equations. The answer for the case of the CF hierarchy of equations and more precisely for their Hamiltonian structures and conservation laws is affirmative. We have obtained the same Poisson tensors P and Q over the manifold of potentials as have been obtained using the hierarchies of Poisson structures over elliptic algebras, see [32]. Hence, we have the same hierarchy of equations and the same conservation laws for it. However, there are still many questions arising. The results we are citing are obtained using a slightly different technique and in order to present them we must introduce the so-called elliptic algebras. The elliptic algebra $\mathcal{G}_e$ used for describing the CF hierarchy of equations and their Hamiltonian structures is defined by the generators

$$\begin{aligned} &X_\alpha^{2l+1} = \omega^{2l}\omega_\alpha X_\alpha \\ &X_\alpha^{2l+2} = \omega^{2l}\omega_\alpha^{-1}\omega_1\omega_2\omega_3 X_\alpha \\ &l \in \mathbb{Z} \qquad \alpha = 1, 2, 3. \end{aligned} \tag{90}$$

In the above formulae X_α are the generators of $so(3) \sim su(2)$ with commutation relations

$$[X_\alpha, X_\alpha] = \epsilon_{\alpha\beta\gamma} X_\gamma$$

and $\epsilon_{\alpha\beta\gamma}$ is the three-dimensional Levi–Civita symbol. As a rule, in the literature the generators X_α are expressed through the Pauli matrices. The quantities $\omega, \omega_\alpha, \alpha = 1, 2, 3$ satisfy the quadratic relations

$$\omega_\alpha^2 - \omega_\beta^2 = r_\beta - r_\alpha \qquad \omega^2 - \omega_\alpha^2 = d_\alpha \equiv r_\alpha - \tfrac{1}{3}(r_1 + r_2 + r_3) \tag{91}$$

where $r_\alpha = -j_\alpha^2$. Usually the following natural parametrization for ω, ω_α is used:

$$\omega_\alpha = \sqrt{\mathcal{P}(\lambda) - d_\alpha} \qquad \omega = \sqrt{\mathcal{P}(\lambda)} \tag{92}$$

where $\mathcal{P}(\lambda)$ is the Weirstrass function: $(\mathcal{P}')^2 = 4(\mathcal{P} - d_1)(\mathcal{P} - d_2)(\mathcal{P} - d_3)$. We prefer the expressions through the Weirstrass function rather than the usual expressions with the Jacobi elliptic functions, but of course it is all the same. One has the following commutation relations of the generators (90) defining the so-called elliptic algebra:

$$\begin{aligned}
&[X_\alpha^{2l}, X_\beta^{2m}] = \epsilon_{\alpha\beta\gamma}(X_\gamma^{2(l+m)} - d_\gamma X_\gamma^{2(l+m-1)}) \\
&[X_\alpha^{2l}, X_\beta^{2m+1}] = \epsilon_{\alpha\beta\gamma}(X_\gamma^{2(l+m)+1} - d_\gamma X_\gamma^{2(l+m)-1}) \\
&[X_\alpha^{2l+1}, X_\beta^{2m+1}] = \epsilon_{\alpha\beta\gamma} X_\gamma^{2(l+m+1)}.
\end{aligned} \tag{93}$$

These relations are consequences both from the commutation relations of $su(2)$ and the properties of the elliptic functions.

The tensors P and Q we have used in the above arise as restrictions over some submanifold [32], of the natural Poisson–Lie tensors for the elliptic algebra $\mathcal{G}_e$ and its central extension with the help of the Gel'fand–Fuchs cocycle.

Comparing the two approaches we note that all the results we have obtained in this paper can also be formulated in terms of graded algebras. For example, in our approach we actually use the graded algebra $\mathcal{G}_p$ generated by the elements

$$\begin{aligned}
&N_\alpha^n = -\tfrac{1}{2}\lambda^n \{e_\alpha\}_I(\lambda + J) \\
&M_\alpha^n = -\tfrac{1}{2}\lambda^n \{e_\alpha\}_{II}(\lambda + J) \\
&n \in \mathbb{Z} \qquad \alpha = 1, 2, 3
\end{aligned} \tag{94}$$

where $e_\alpha, \alpha = 1, 2, 3$ is the usual orthogonal basis in $\mathbb{R}^3$, that is $(e_\alpha)_\beta = \delta_{\alpha\beta}$. The commutation relations between this generators are

$$\begin{aligned}
&[N_\alpha^n, N_\beta^m] = \epsilon_{\alpha\beta\gamma}(N_\gamma^{n+m+1} + j_\gamma M_\gamma^{n+m}) \\
&[M_\alpha^n, M_\beta^m] = \epsilon_{\alpha\beta\gamma}(M_\gamma^{n+m+1} + j_\gamma N_\gamma^{n+m}) \\
&[N_\alpha^n, M_\beta^m] = -\epsilon_{\alpha\beta\gamma}(j_\beta N_\gamma^{n+m} + j_\alpha M_\gamma^{n+m}).
\end{aligned} \tag{95}$$

We must stress that in writing the above relations *we used implicitly the new Lie algebra structure that we introduced over the algebra so(4) and the above relations are not simple consequences from the usual Lie algebra structure over so(4).*

Then again the tensors P and Q can be obtained by restricting the natural Poisson–Lie tensors for $\mathcal{G}_p$ and its central extension over some submanifold. One easily finds that the resulting Poisson submanifolds used in both constructions are isomorphic and then the two approaches based on the elliptic algebra $\mathcal{G}_e$ and on the algebra $\mathcal{G}_p$ are equivalent. One can then imagine that these algebras are isomorphic but until now we had failed to prove it. Thus, we have the 'experimental' result that the two algebras $\mathcal{G}_e$ and $\mathcal{G}_p$ generate the same Poisson structures over some submanifolds but it is still an open question why this occurs.

Acknowledgments

The authors would like to thank the Institute for Mathematics at the Leipzig University for their kind hospitality during his stay in 1996 where part of this paper was done. This work was supported by contract N 141/96 with the St Kliment Ohridski University, Sofa, Bulgaria, by contract MM 608/96 with the Ministry of Science Education and Technologies of Bulgaria, by the DAAD (Germany) and by the NTZ of Leipzig University.

References

[1] Bordag L A and Yanovski A B 1995 *J. Phys. A: Math. Gen.* **28** 4007
[2] Bordag L A and Yanovski A B 1996 *J. Phys. A: Math. Gen.* **29** 5575
[3] Borovik A E 1978 *Lett. JETP.* **28** 629
Borovik A E *Lett. JETP.* 1978 **35** 36 873
[4] Cherednik I V 1981 *Nucl. Phys.* **33** 278
[5] Drinfeld V G and Sokolov V V 1984 Lie algebras and Korteweg–de Vries type equations *VINITI, Contemporary Problems of Mathematics. Recent Developments* vol 24 (Moscow) p 81
[6] Faddeev L D and Takhtadjan 1987 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer)
[7] Barouch E, Fokas A S and Pageorgiou V G 1987 *Preprint* Clarkson University, NY
[8] Barouch E, Fokas A S and Papageorgiou V G 1988 *J. Math. Phys.* **29** 2628
[9] Frölicher A and Nijenhuis A 1965 *Ned. Acad. Westinch. Proc.* A **59** 338
[10] Fuchssteiner B 1984 *Physica* **13D** 387
[11] Gel'fand I M and Dorfman I Y 1979 *Funct. Anal. Appl.* **13** 13
[12] Gel'fand I M and Dorfman I Y 1980 *Funct. Anal. Appl.* **14** 71
[13] Gerdjikov V S and Yanovski A B 1984 *Phys. Lett.* **103A** 232
[14] Gerdjikov V S and Yanovski A B 1984 *Commun. Math. Phys.* **103** 549
[15] Gerdjikov V S and Yanovski A B 1994 *J. Math. Phys.* **35** 3687
[16] Goto M and Grosshans F 1978 Semisimple Lie algebras *Lecture Notes in Pure and Applied Mathematics* vol 38 (New York: Dekker)
[17] Kirillov A A 1962 Unitary representations of nilpotent Lie groups *Russ. Math. Surv.* **17** 53–104
[18] Landau L D and Lifshitz E M 1965 *Collected Papers of L D Landau* ed D ter Haar (New York: Pergamon, Gordon and Breach)
[19] Lichnerovich A 1975 New geometrical dynamics *Differential Geometrical Methods in Mathematical Physics (Lect. Notes Math. **570**)* ed A Dold and B Eckmann (New York: Springer)
[20] Lie S 1890 *Theorie der Transformationsgruppen, Zweiter Abschnitt, Unter Mitwirkung von. Prof. Dr. Frederich Engel* (Leipzig: Teubner)
[21] Magri F 1968 *J. Math. Phys.* **19** 1156
[22] Magri F 1980 A geometrical approach to the nonlinear solvable equations *Lecture Notes in Physics* vol 120 (New York: Springer)
[23] Magri F and Morosi 1968 *Preprint* Dipartamento di Mathematica, Universita di Milano
[24] Magri F, Morosi C and Ragnisco O 1985 *Commun. Math. Phys.* **99** 115
[25] Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetry* (Berlin: Springer)
[26] Marsden J E and Ratiu T S 1986 *Lett. Math. Phys.* **11** 161
[27] Mikhalev V G 1988 *Z. LOMI* **172** 99
[28] Reymann A G 1980 *Z. LOMI* **95** 3
[29] Reymann A G 1983 *Z. LOMI* **131** 118
[30] Reymann A G and Semenov-Tian Shanski M A 1980 *Dokl. Acad. Nauk USSR* **251** 1310
[31] Reymann A G and Semenov-Tian Shanski M A 1980 *Funct. Anal. Appl.* **14** 77–8
[32] Sidorenko U N 1987 *Z. LOMI* **161** 76
[33] Sklianin A E 1979 *Preprint* E-3-79, Steklov Mathamatical Institute LOMI
[34] Souriau J M 1970 *Structure des Systèmes Dynamiques* (Paris: Dunod)
[35] Stone A P 1969 *J. Diff. Geom.* **3** 447
[36] Stone A P 1973 *Can. J. Math.* **25** 903
[37] Zakharov V E and Takhtadjan 1979 *Teor. Mat. Fiz.* **38** 26